Complexified moduli of non-commutative instantons

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Abstract:
We revisit the generalised ADHM construction for instantons in non-commutative space using a manifestly quaternionic formalism. This leads to an identification of the self-dual part of $\theta^{\mu\nu}$ as the imaginary part of the size modulus of the instanton.

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1. Introduction

In this note, we would like to demonstrate that the non-commutative resolution of the instanton moduli space can be viewed as due to a complexification of the size modulus. This achieved by reexpressing the component construction of non-commutative instantons in the quaternionic formalism of [CSW] that mirrors the realisation of instantons via the Hopf fibration of S^7.

We find that the self-dual part of θ, the measure of non-commutativity, is an obstruction to solving the ADHM equation and that one has to generalise the construction to include complexified quaternions. In the end, however, the components of the physical gauge fields are still real, but, as anticipated from other arguments, the non-commutative instanton does not fit into a $SU(2)$ but only into a $U(2)$.

The structure of this note is as follows: First we review the quaternionic construction in the classical, commutative setting in order to establish our notation. Furthermore, this allows us to highlight the way different assumptions enter the construction and show where the non-commutative generalisation differs. The next section discusses this generalisation in depth and translates the quaternionic language to the component language found in the existing literature on non-commutative instantons.

2. The commutative construction

In a physically sensible gauge theory, the gauge field should have real components as the imaginary parts of complex gauge fields would have a kinetic term that would not be bounded from below. Usually, this restriction is automatically dealt with by using a a compact gauge group and a real formalism in which no explicit factors of i appear. However, in the case of non-commutative coordinates, this is more difficult as commutators of coordinates contain factors of i. In the simplest case of the non-commutative plane that we will be concerned with in this note, the basic commutator is given by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}. \tag{2.1}$$

Here $\theta^{\mu\nu}$ is a real, anti-symmetric matrix. As in quantum mechanics, the factor of i is needed as the coordinates should be promoted to *hermitian* operators and the commutator of such is anti-hermitian.

In order to make reality (or anti-hermiticity of the connection) explicit, we will re-express the non-commutative version[NS] of the ADHM construction[ADHM] in the quaternionic formalism of [CSW], in which the anti-hermiticity is manifest.

Let us start by reviewing this construction in the case of commuting coordinates. Later we will then contrast this with the non-commutative version. The quaternionic formalism is based on the identification

$$\mathbb{R}^4 \quad \text{(the space)} \quad \leftrightarrow \quad u(2) \quad \text{(the gauge algebra)} \quad \leftrightarrow \quad \mathbb{H} \quad \text{(the quaternions)}$$

where the first arrow relates two vector spaces but the second holds not only in the Lie algebra (with commutators) but also in the matrix (enveloping) algebra.

The quaternions $\mathbb{H}$ are generated as a real algebra by σ_1, σ_2 and σ_3 with the relations $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -1$ and

$$\sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = \sigma_3 \qquad \text{and cyclic.}$$

It is convenient to define $\sigma_4 = 1$ and then each $q \in \mathbb{H}$ can be written as $q = q^\mu \sigma_\mu = q^4 + q^1 \sigma_1 + q^2 \sigma_2 + q^3 \sigma_3$. Under quaternionic conjugation, q is mapped to $\bar{q} = q^\mu \bar{\sigma}_\mu = q^4 - q^1 \sigma_1 - q^2 \sigma_2 - q^3 \sigma_3$. The imaginary (or anti-hermitian) are closed under taking commutators and will be identified with $su(2) \subset u(2)$. Note that

$$\bar{q}q = q_1^2 + q_2^2 + q_3^2 + q_4^2 = \|q\|^2$$

is real and therefore commutes with all quaternions.

Now we can describe the commutative ADHM construction. For simplicity we will restrict ourselves to the case of $su(2) = sp(1)$ gauge theory. This starts with picking a $(k+1) \times k$ quaternionic matrix M that is linear in x, so that we can write it as

$$M = B + Cx.$$

Note that Cx is the product of a matrix with a (quaternionic) scalar and thus is again a matrix rather than a vector as a result of an inner product of a matrix and a vector! We have to require that the matrix $R = M^\dagger M$ is in $GL(k, \mathbb{R})$ for all x, that is, it is real and invertible. This requirement of reality is what is usually referred to as "the ADHM equations" as it requires the three imaginary parts of the matrix R to vanish.

Next, we observe that the rank of M can maximally be k, thus there is a non-vanishing x-dependent vector $N \in \mathbb{H}^{k+1}$ that is annihilated by $M^\dagger$ (defined as the transpose of the quaternionic conjugate of M):
$$M^\dagger N = 0 \tag{2.2}$$

Furthermore, we can require N to be of unit length:

$$N^\dagger N = 1 \tag{2.3}$$

With this, we can define a gauge connection as $A_\mu = N^\dagger \partial_\mu N$ that has a self-dual field-strength with instanton number k.

Let us check these properties explicitly, so it becomes clear where the various assumptions enter. First observe that the anti-hermiticity of A_μ directly follows from taking ∂_μ of the normalisation condition (2.3). The field-strength is

$$F_{\mu\nu} = \partial_\mu A_\nu + A_\mu A_\nu - \mu \leftrightarrow \nu = (\partial_\mu N^\dagger)\partial_\nu N + N^\dagger(\partial_\mu N)N^\dagger(\partial_\nu N) - \mu \leftrightarrow \nu.$$

Now, we use again the derivative of (2.3) to bring $F_{\{\mu\nu}$ to the form $(\partial_\mu N^\dagger)(\mathbb{1}_{k+1} - NN^\dagger)(\partial_\nu N)$. From the definition of N it follows that $\mathbb{1}_{k+1} - NN^\dagger$ is a projector on the subspace where $MM^\dagger$ is invertible or, put differently, $MR^{-1}M^\dagger + NN^\dagger$ is the unit matrix $\mathbb{1}_{k+1}$. Using the derivative of (2.2), we arrive at

$$F_{\mu\nu} = (\partial_\mu N^\dagger)MR^{-1}M^\dagger(\partial_\nu N) - \mu \leftrightarrow \nu = N^\dagger(\partial_\mu M)R^{-1}(\partial_\nu M^\dagger)N - \mu \leftrightarrow \nu.$$

As M is linear in x, we have $\partial_\mu M = C\sigma_\mu$. Finally we have to use the fact that R is real and therefore commutes with σ_μ to arrive at

$$F_{\mu\nu} = N^\dagger CR^{-1}\sigma_{[\mu}\bar\sigma_{\nu]}C^\dagger N.$$

From the defining relation of the imaginary units we see

$$\sigma_1\bar\sigma_2 = -\sigma_3 = -\sigma_3\bar\sigma_4$$

and cyclic which implies that $\sigma_{[\mu}\bar\sigma_{\nu]}$ is anti-self-dual which concludes the proof of the anti-self-duality of F. Similarly, $\bar\sigma_{[\mu}\sigma_{\nu]}$ is self-dual, and thus by baring the above construction we would construct anti-instantons.

As a next step, let us do this construction explicitly for $k = 1$, the case of a single instanton. We start with

$$M = \begin{pmatrix} b_1 + c_1 x \\ b_2 + c_2 x \end{pmatrix}.$$

By shifting the origin in x-space we can assume $b_2 = 0$. Thinking of M as being a linear map $M: \mathbb{H} \to \mathbb{H}^2$, we can choose an adapted basis of $\mathbb{H}$ and $\mathbb{H}^2$ such that in this basis $c_1 = 0$ and $c_1 = 1$. After these changes of coordinates

$$M = \begin{pmatrix} b \\ x \end{pmatrix}.$$

For this ansatz, we have to check the ADHM equations. We compute

$$M^\dagger M = \|b\|^2 + \|x\|^2 \tag{2.4}$$

which is automatically real and everywhere invertible if $b \neq 0$. Next, we have to solve for $N = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. From (2.2), we find

$$0 = \bar b n_1 + \bar x n_2 \quad \Rightarrow \quad n_1 = -\frac{b\bar x n_2}{\|b\|^2}.$$

Similarly, (2.3) implies

$$\|n_2\|^2 = \frac{\|b\|^2}{\|x\|^2 + \|b\|^2}.$$

From this, we obtain

$$N = \frac{\|b\|}{\sqrt{\|x\|^2 + \|b\|^2}} \begin{pmatrix} -\frac{b\bar{x}}{\|b\|^2} \\ 1 \end{pmatrix} \lambda \tag{2.5}$$

for some x dependent $\lambda \in \mathbb{H}$ with $\|\lambda\| = 1$. Finally, we can compute the connection:

$$A_\mu = \bar{\lambda} \frac{x_\mu + x\bar{\sigma}_\mu}{\|x\|^2 + \|b\|^2} \lambda + \bar{\lambda} \partial_\mu \lambda,$$

from which we recognise λ as a gauge parameter (unit norm is equivalent to the restriction $su(2) \subset u(2)$) and up to a gauge transformation

$$A_\mu = \frac{x_\mu + x\bar{\sigma}_\mu}{\|x\|^2 + \|b\|^2}.$$

After doing a rotation in quaternion space we can assume $b \in \mathbb{R}_{>0}$ and we recognise this as the size modulus of the instanton. The four position moduli have been fixed above by setting $b_2 = 0$. This concludes our review of the quaternionic ADHM construction.

3. The non-commutative ADHM-construction

In the non-commutative case, we can try to apply the same strategy as in the commutative case. There is only one important difference: Let us compute $\bar{x}x$.

$$\bar{x}x = x^\mu x^\nu \bar{\sigma}_\mu \sigma_\nu = x^{[\mu} x^{\nu]} \bar{\sigma}_\mu \sigma_\nu + x^\mu x^\nu \bar{\sigma}_{(\mu} \sigma_{\nu)}.$$

For the first term, we use (2.1) and recall that the combination $\bar{\sigma}\sigma$ is self-dual, while for the second we use $\bar{\sigma}_{(\mu}\sigma_{\nu)} = \delta_{\mu\nu}$ which follows from the definition of quaternions. So we arrive at

$$\bar{x}x = 2i\theta_{SD}^{\mu\nu} \bar{\sigma}_{[\mu}\sigma_{\nu]} + \|x\|^2.$$

It turns out that this is in fact the only direct difference of the non-commutative construction to the one of the previous section.

As an immediate observation we recover the well known fact that only the self-dual part of θ contributes. In fact, in the case of an anti-self-dual background, none of the formulae change.

The second important observation is that the self-dual matrix $\bar{\sigma}_{[\mu}\sigma_{\nu]}$ changes sign under quaternionic conjugation. It is thus purely imaginary! This appears to pose an obstruction to the ADHM equation (2.4) if we try to construct a non-commutative instanton:

$$M^\dagger M = \|b\|^2 + \|x\|^2 + 2i\theta_{SD}^{\mu\nu} \bar{\sigma}_{[\mu}\sigma_{\nu]} \in^? GL(1, \mathbb{R}) = \mathbb{R}^*$$

Two of the three terms are manifestly real while the third is purely imaginary (in the quaternionic sense that prevents it to commute with other quaternions as discussed above;

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the i does not play a role her) and cannot be cancelled to yield a real sum. To our knowledge, this obstruction has not been discussed in the literature on non-commutative instantons, mainly due to the fact that it is manifest only in the quaternionic formalism.

The reader might be worried[BSST] that what we have described here is really the construction of an instanton in $sp(2)$ rather than $su(2)$ gauge theory and that although these theories are the same in the commutative setting they might differ in non-commutative space. Indeed, applying the rules[CSW] for the ADHM construction for the $su(N)$ series requires M to be a quaternionic 4×1 matrix for which in addition to the above ADHM condition also

$$M^\dagger \sigma_1 M$$

has to be real. However, it is not hard to check that up to conjugation in the quaternions this additional condition is solved by

$$M = \begin{pmatrix} b \\ \sigma_2 b \\ x \\ \sigma_2 x \end{pmatrix}.$$

With this ansatz, the rest of the construction reduces to the case discussed above.

The existing literature constructs instantons in non-commutative space without encountering the problem of this imaginary contribution to the ADHM construction. Let us therefore translate our findings to the conventional component formalism and see how the obstruction is circumvented there. To be concrete, we will compare to the formalism used in [CKT], but then making contact with other treatments of the subject, for example [N][DN][KLY] should be straight forward. To do so we use the following representation of the quaternionic units in terms of 2×2 matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The reader should not be confused that although some of these matrices have complex entries, we are still dealing with a real algebra, that is the general element of $\mathbb{H}$ is a linear combination of these matrices with *real* coefficients. For example, the quaternion that we denote x is in this matrix notation

$$x = \begin{pmatrix} x^4 + ix^3 & x^1 + ix^2 \\ -x^1 + ix^2 & x^4 - ix^3 \end{pmatrix}.$$

Furthermore, we choose coordinates that skew diagonalise $\theta^{\mu\nu}$:

$$\theta^{\mu\nu} = \begin{pmatrix} & \theta_1 & & \\ -\theta_1 & & & \\ & & & \theta_2 \\ & & -\theta_2 & \end{pmatrix}.$$

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So the purely imaginary quaternion becomes

$$2i\theta^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu = 4i(\theta_1 + \theta_2)\sigma_3 = \begin{pmatrix} -4(\theta_1 + \theta_2) & \\ & 4(\theta_1 + \theta_2) \end{pmatrix}. \tag{3.1}$$

In order to fulfil the ADHM equations, this contribution should be cancelled by a term from b. [CKT] do this (in their equation (6.11) adopted to our conventions) by taking b to be the matrix

$$b = \begin{pmatrix} & \rho \\ \sqrt{8(\theta_1 + \theta_2) + \rho^2} & \end{pmatrix} \tag{3.2}$$

and indeed

$$\bar{b}b = \begin{pmatrix} & \rho \\ \sqrt{8(\theta_1 + \theta_2) + \rho^2} & \end{pmatrix}^\dagger \begin{pmatrix} & \rho \\ \sqrt{8(\theta_1 + \theta_2) + \rho^2} & \end{pmatrix}$$

$$= \begin{pmatrix} \rho^2 + 4(\theta_1 + \theta_2) & \\ & \rho^2 + 4(\theta_1 + \theta_2) \end{pmatrix} + \begin{pmatrix} 4(\theta_1 + \theta_2) & \\ & -4(\theta_1 + \theta_2) \end{pmatrix}.$$

So, obviously $\bar{x}x + \bar{b}b$ is proportional to the unit matrix as required by the ADHM equation and thus real when translated back to quaternions.

How does this solution fit into the quaternionic framework? To answer this question, we have to translate the ansatz (3.2) by expressing it as a linear combination of the σ_μ:

$$b = \left(\frac{\rho}{2} - \sqrt{2(\theta_1 + \theta_2) + \frac{\rho^2}{4}}\right)\sigma_1 - i\left(\frac{\rho}{2} + \sqrt{2(\theta_1 + \theta_2) + \frac{\rho^2}{4}}\right)\sigma_2.$$

The important difference to what we tried above is that because of the i this is not a real linear combinations of the σ_μ but an element of the complexified quaternions! So, to solve the ADHM equations in a self-dual background, we have to allow for complex components. This together with the matrix inspired conjugation rule

$$\bar{q} = -\bar{q}^1\sigma_1 - \bar{q}^2\sigma_2 - \bar{q}^3\sigma_3 + \bar{q}^4$$

where the bar on the coefficients is complex conjugation makes $\bar{q}q$ an arbitrary (ordinary, not complexified) quaternion:

$$\bar{q}q = \|q\|^2 + 2i\Im(\bar{b}^\mu b^\nu)\bar{\sigma}_{[\mu}\sigma_{\nu]}$$

Once we have made the generalisation to complexified quaternions we see that we could conjugate b to a more conventional form and parametrise the solutions of the ADHM equations as

$$b = \eta - \frac{i}{\eta}\theta^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu, \tag{3.3}$$

as this leads to

$$\bar{b}b = \eta^2 + \frac{\theta_{SD}^2}{\eta^2} - 2i\theta^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu$$

and cancels the imaginary part of $\bar{x}x$.

In this expression we can interpret the real part as the square of the size of the instanton (corresponding to $\|b\|^2$ in the commutative case) and we recover the well known result that in the presence of a non-commutativity with a self-dual component, the minimal size of instantons is $\sqrt{2\theta_{SD}} > 0$.

One might be worried that the complexification of the components of M and thus N would complexify the components of the physical field A_μ as well and thus thus render the non-commutative instantons unphysical as negative kinetic terms arrise for non-compact (for example complexified) gauge groups. However, the effect of the complexification of M is still physically acceptable: Once again taking the derivative of (2.3) (where now the adjoined includes the conjugation on the complexified quaternions), one finds

$$A_\mu = N^\dagger \partial_\mu N$$

to be odd under conjugation (as in the commutative case). If A_μ would be an ordinary quaternion this would imply that its real part vanishes and A_μ is in $su(2)$. In the complexified case however, it just means the coeffiecients of the $su(2)$ generators in A_μ are real and the coefficient of the $u(1) \subset u(2)$ is imaginary. But in our convention this means that that A_μ is in the compact real form of $u(2)$. The effect of this generalization to complexified quaternions is just that the non-commutative instantons do not fit in $su(2)$. One has to enlarge the gauge group to $u(2)$, as it is well known for non-commutative gauge theories.

However, the main result of applying the quaternionic formalism and finding the need for the complexification is (3.3), which shows that what used to be the real size modulus in the commutative case has now been complexified and θ_{SD}, the self-dual component of the background, appears as its imaginary part. This fits well into the general pattern of how string theory avoids singularities in moduli spaces: There is a real geometric modulus for which the theory becomes singular at the origin. But string theory pairs this real modulus together with the flux of a background field into a complex modulus so the singularity at the origin can be avoided by turning on that flux and thus going into the complex plane.

4. Discussion

We translated the generalisation of the ADHM construction of instantons to non-commutative spaces to the quaternionic formalism. We found that for $\theta_{SD} \neq 0$ it is not possible to solve the ADHM equations over the quaternions. Rather we had to allow for quaternions with complex coefficients.

From this point on, all expressions have complex coefficients in principle, but we found that in the case of the gauge field A_μ which is a physical field as opposed to M and N, anti-hermiticity and thus compactness of the gauge group and positivity of the kinetic term are all preserved. The only effect of the complexification is that the trace will no longer vanish and thus the gauge field is in $U(2)$ rather than $SU(2)$.

However, there remain a couple of open questions: In the commutative case, we knew that N is nowhere vanishing and thus could trivially impose the normalisation (2.3). For non-commutative $*$-multiplication however, there is in general no $*$-division. Still one can calculate inverses in the operator sense, at least for operators of the form $1 - O$, the geometric series gives inverses at least for operators not having a spectrum which intersects the unit circle.

By the nature of these operator inverses, explicit solutions are not available and thus we only have implicit knowledge of the physical profile of non-commutative instantons.

Furthermore, for functions like M and N that depend on x, even real expressions do not commute anymore and one has to be extremely careful with orderings like in expressions of the form of (2.5).

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